APPROXIMATE SOLUTION OF A SINGULAR INTEGRAL EQUATION BY MEANS OF JACOBI POLYNOMIALS

(PRIBLIZHENNOE RESHENIE ODNOGO SINGULIARNOGO INTEGRAL'NOGO URAVNIENIIA PRI POMOSHCHI MNOGOCHLENOV IAKOBI)

PMM Vol. 30, No. 3, 1966, pp. 564-569

L.N. KARPENKO
(Novosibirsk)

(Received May 3, 1965)

Many problems of great practical value can be reduced to one-dimensional singular integral equations with constant coefficients, on the open curve.

Amongst them we can, for example, find the problem of a plane flow around the arc, the problem of propagation of a crack in a homogeneous or non-homogeneous elastic plate, contact problems of the plane theory of elasticity, some problems of rock mechanics e.a.

In each of these cases, the behavior of the unknown function near the ends of the curve is found to be the characteristic feature of the problem. Orthogonal polynomials the weight of which is given by the canonical function of the equation, the function determining the behavior of the solution near the ends, are found convenient to use in the approximate solutions of each singular equations.

If the coefficients of the equation are constant, then these polynomials are found to be Jacobi polynomials.

Examples of application of orthogonal polynomials in the solutions of singular integral equations are given in [1 and 2], while [3] uses the trigonometric form of Chebyshev polynomials and their analogs.

In the present paper a method of solution of a singular integral equation based on the properties of Jacobi polynomials, is given. The equation has constant coefficients on the segment $(-1, +1)$ of the real axis and different, separate conditions at each end of this segment.

1. We shall begin by quoting some already known (see e.g. [4]) facts concerning the solution of the characteristic equation

$$a\varphi(x) + \frac{b}{\pi i}\int_{-1}^{1}\frac{\varphi(t)}{t-x}\,dt = f(x) \qquad (-1 < x < 1) \tag{1.1}$$

with constant coefficients a and b and with the right-hand side $f(x)$, satisfying Hőlder's condition (condition H or $H(\mu)$)

$$|f(t)-f(x)| \leqslant A\,|t-x|^{\mu} \qquad (0<\mu\leqslant 1) \qquad (t,\, x\in[-1,+1]) \tag{1.2}$$

We assume that the principal value in Cauchy's sense of the integral in (1.1) is considered and, that additional conditions

$$b\neq 0, \qquad a^2-b^2=1, \qquad 0\leqslant \theta=\arg(a-b)<\pi$$

are fulfilled.

We shall seek the solution of (1.1) in the class of functions satisfying the condition H on the interior points and admitting the integrable infinity at the ends of the interval of integration (in the class H^* using the nomenclature of Muskhelishvili [4]). Let us put $a-b=\rho e^{i\theta}$, and introduce

$$\alpha=-\lambda-\varkappa+\frac{\theta}{\pi}+\frac{\ln\rho}{\pi i}, \qquad \beta=\lambda-\frac{\theta}{\pi}-\frac{\ln\rho}{\pi i} \tag{1.3}$$

The above conditions define λ and $\varkappa$ uniquely with one exception (when $\theta=0$, then $\lambda=\varkappa=\operatorname{Re}\alpha=\operatorname{Re}\beta=0$, at the same time $\operatorname{Im}\alpha=\operatorname{Im}\beta\neq 0$). A detailed treatment of all the possible cases is given in [4]. In the present problem, $\varkappa=-\alpha-\beta$ (index of equation) can assume the values of -1, 0 and $+1$. In these cases the solution of (1.1) is

$$\varphi(x)=af(x)-\frac{bw(x)}{\pi i}\int_{-1}^{1}\frac{f(t)}{w(t)}\frac{dt}{t-x}+Cw(x) \qquad (-1<x<1) \tag{1.4}$$

where C is an arbitrary constant when $\varkappa=1$, and zero otherwise.

A branch of the function

$$w(x)=(1-x)^{\alpha}(1+x)^{\beta} \qquad (-1<x<1) \tag{1.5}$$

will, for convenience, be defined by the condition $w(0)=1$. If $\varkappa=-1$, then we must supplement (1.4) with the condition

$$Uf\equiv\int_{-1}^{1}\frac{f(x)}{w(x)}\,dx=0 \tag{1.6}$$

which is also the necessary condition for the solution of (1.1) to exist.

2. We shall use Rodrigues formula [5 and 6]

$$w(x)P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}[w(x)(1-x^2)^n] \qquad (n=0,1,2,\ldots) \tag{2.1}$$

The following relations are also true

$$\int_{-1}^{1}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)w(x)\,dx=0 \qquad (m\neq n)$$

$$\int_{-1}^{1}[P_n^{(\alpha,\beta)}(x)]^2 w(x)\,dx=h_n^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)} \tag{2.2}$$

In the following $\alpha+\beta=-\varkappa$, $\varkappa$ is a whole number and $\varkappa\geqslant -1$ (although in (2.1) we can consider α and β to be arbitrary, while in (2.2) we assume that $\operatorname{Re}\alpha > -1$, $\operatorname{Re}\beta > -1$).

Let $w(z)$ be an analytic continuation of (1.5) into the complex plane $z = x + iy$ with a cut along the real axis, extending from $z = -1$ to $z = +1$ so, that $w(z) = w(x + i0)$. Using the Rodrigues formula (or simply the formulas for differentiating and integrating Jacobi polynomials [5]), we shall obtain the asymptotic expansion of this function

$$w(z) = e^{-i\pi\alpha}2^{-\varkappa}P_{-\varkappa}^{(-\alpha,\,-\beta)}(z) + e^{-i\pi\alpha}\sum_{k=1}^{\infty} 2^{k-\varkappa} P_{k-\varkappa}^{(-\alpha-k,\,-\beta-k)}(0)\frac{1}{z^k} \tag{2.3}$$

Here and in the following, $P_n^{(\alpha,\beta)}(z) \equiv 0$, if $n < 0$. From (2.3), we have

$$(1-z^2)^n w(z) = (1-z)^{n+\alpha}(1+z)^{n+\beta} = (-1)^n e^{-i\pi\alpha} 2^{2n-\varkappa} P_{2n-\varkappa}^{(-\alpha-n,\,-\beta-n)}(z) + o(1) \tag{2.4}$$

It can be shown that

$$\frac{1}{\pi}\int_{-1}^{1}\frac{w(t)}{t-z}\,dt = \frac{e^{i\pi\alpha}}{e^{i\pi\alpha}-e^{-i\pi\alpha}}\,\frac{1}{\pi}\int_{\Gamma}\frac{w(\zeta)}{\zeta - z}\,d\zeta \qquad (z \,\overline{\in}\, [-1,\,+1])$$

Here, the integration in the right-hand side is performed along an arbitrary contour, provided it encloses the points -1 and $+1$, and leaves the point z outside. Then, from (2.3), we can obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{w(t)}{t-z}\,dt = \frac{1}{\sin\pi\alpha}\left[e^{i\pi\alpha} w(z) - 2^{-\varkappa}P_{-\varkappa}^{(-\alpha,\,-\beta)}(z)\right] \qquad (z \,\overline{\in}\, [-1,\,1])$$

which, together with (2.1) and (2.4) gives, after the n-tuple integration by parts followed by differentiation, the integral

$$\frac{1}{\pi}\int_{-1}^{1}\frac{P_n^{(\alpha,\,\beta)}(t)}{t-z}\,w(t)\,dt = \frac{1}{\sin\pi\alpha}\left[e^{i\pi\alpha}P_n^{(\alpha,\,\beta)}(z)\,w(z) - 2^{-\varkappa}P_{n-\varkappa}^{(-\alpha,\,-\beta)}(z)\right] \tag{2.5}$$

The latter can also be obtained by another method, in which the Jacobi polynomial and Jacobi functions of the second kind are given in terms of hypergeometric functions. (2.5) yields the principal value of the integral

$$\frac{1}{\pi}\int_{-1}^{1}\frac{P_n^{(\alpha,\,\beta)}(t)}{t-x}\,w(t)\,dt = \cot\pi\alpha P_n^{(\alpha,\,\beta)}(x)\,w(x) - \frac{2^{-\varkappa}}{\sin\pi\alpha}P_{n-\varkappa}^{(-\alpha,\,-\beta)}(x) \qquad (-1 < x < 1) \tag{2.6}$$

Hence, utilising the relationship between α and β (1.3) and the coefficients of (1.1), we obtain

$$aP_n^{(\alpha,\,\beta)}(x)\,w(x) + \frac{b}{\pi i}\int_{-1}^{1}\frac{P_n^{(\alpha,\,\beta)}(t)}{t-x}\,w(t)\,dt = (-1)^{\lambda+\varkappa}2^{-\varkappa}P_{n-\varkappa}^{(-\alpha,\,-\beta)}(x) \tag{2.7}$$

$$a\,\frac{P_n^{(-\alpha,\,-\beta)}(x)}{w(x)} - \frac{b}{\pi i}\int_{-1}^{1}\frac{P_n^{(-\alpha,\,-\beta)}(t)}{t-x}\,\frac{dt}{w(t)} = (-1)^{\lambda+\varkappa}2^{\varkappa}P_{n+\varkappa}^{(\alpha,\,\beta)}(x) \qquad (-1 < x < 1) \tag{2.8}$$

3. We shall continue the investigation of (1.1). Let us introduce another function $\psi(x)$ and an operator S, and let both be given by

$$\varphi(x)=w(x)\psi(x),\quad S\psi=aw(x)\psi(x)+\frac{b}{\pi i}\int_{-1}^{1}\psi(t)\frac{w(t)}{t-x}\,dt \tag{3.1}$$

We shall write (1.1) as

$$S\psi=f \tag{3.2}$$

Its solution will, according to (1.4), be

$$\psi^*=Rf,\qquad Rf=a\frac{f(x)}{w(x)}-\frac{b}{\pi i}\int_{-1}^{1}\frac{f(t)}{w(t)}\frac{dt}{t-x} \tag{3.3}$$

At the same time, the arbitrary constant in (1.4) will, for $\varkappa=1$, be fixed by the condition

$$\int_{-1}^{1}\psi^*(x)w(x)\,dx=0 \tag{3.4}$$

If $\varkappa=-1$, then the additional condition (1.6) will have to be fulfilled. Let us now try to solve the complete singular integral equation

$$S\psi+k\psi=f,\qquad \psi k=\int_{-1}^{1}k(x,t)w(t)\psi(t)\,dt \tag{3.5}$$

The operator S is defined by the formula (3.1); a, b, and $f(x)$ satisfy the previous (par. 1) conditions; α, β and $w(x)$ are given by (1.3) and (1.5), while the kernel $k(x,t)$ satisfies the condition $H(\mu,\nu)$

$$|k(x,t)-k(y,s)|\leqslant A_1|x-y|^{\mu}+A_2|t-s|^{\nu}\qquad(\mu>\operatorname{Re}\alpha,\ \mu>\operatorname{Re}\beta) \tag{3.6}$$

Let us for definiteness, impose the condition (3.4) on the solution ψ^* of this equation for $\varkappa=1$, and

$$U(k\psi^*-f)=0 \tag{3.7}$$

i.e. solvability of (3.5), for $\varkappa=-1$. Let us also replace the exact equation (3.5) with another, which we shall consider as approximate and possessing a special type kernel

$$S\Psi+K\Psi=f+\alpha_{\varkappa} \tag{3.8}$$

$$K\psi=\int_{-1}^{1}K(x,t)w(t)\psi(t)\,dt,\quad K(x,t)=\sum_{k=0}^{n}N_k(t)P_k^{(-\alpha,-\beta)}(x) \tag{3.9}$$

Here $P_k^{(-\alpha,-\beta)}(x)$ are the Jacobi polynomials, $N_k(t)$ satisfies the H condition for $-1\leqslant t\leqslant 1$; $\alpha_{\varkappa}$ is a constant, $\alpha_0=\alpha_1=0$, and α_{-1} is chosen so, as to secure the existence of the solution of (3.8). Analogously to (3.7), we have

$$\alpha_{-1}=\frac{1}{h_0^{(-\alpha,-\beta)}}U(K\Psi^*-f) \tag{3.10}$$

where $h_0^{(-\alpha,-\beta)}$ is defined by means of (2.2).

Since the kernel $K(x,t)$ is degenerate,

$$K\Psi^* = \sum_{k=0}^{n} a_k P_k^{(-\alpha,\,-\beta)}(x), \qquad a_k = \int_{-1}^{1} N_k(t)\, w(t)\, \Psi^*(t)\, dt \qquad (k = 0, 1, \ldots, n) \tag{3.11}$$

where ψ^* is the solution of (3.8). With this in mind, let us apply (3.3) to (3.8) and utilise (2.8) to obtain

$$\Psi^*(x) = Rf - (-1)^{\lambda+\varkappa} 2^{\varkappa} \sum_{k=-\varkappa}^{n} a_k P_{k+\varkappa}^{(\alpha,\,\beta)}(x) \tag{3.12}$$

If $a_{-1} = 0$ for $\varkappa = 1$ then the solution ψ^* is subject to the condition

$$\int_{-1}^{1} \Psi^*(x)\, w(x)\, dx = 0$$

analogous to (3.4), while for $\varkappa = -1$, it follows from (3.10), that (3.12) will be a solution of (3.8) only, if

$$\alpha_{-1} = a_0 - \frac{Uf}{h_0^{(-\alpha,\,-\beta)}} \tag{3.13}$$

Let us introduce the notation

$$c_{ik} = (-1)^{\lambda+\varkappa} 2^{\varkappa} \int_{-1}^{1} N_i(t)\, P_{k+\varkappa}^{(\alpha,\,\beta)}(t)\, w(t)\, dt \qquad \begin{pmatrix} i = 0, 1, 2, \ldots, n \\ k = -\varkappa, -\varkappa+1, \ldots, n \end{pmatrix}$$

$$b_i = \int_{-1}^{1} N_i(t)\, w(t)\, Rf(t)\, dt \qquad (i = 0, 1, 2, \ldots, n) \tag{3.14}$$

$$c_{-1k} = b_{-1} = 0 \qquad (k = -\varkappa, -\varkappa+1, \ldots, n)$$

Substituting (3.12) into (3.11) we obtain the system of equations determining a_k

$$a_i + \sum_{k=-\varkappa}^{n} c_{ik} a_k = b_i \qquad (i = -\varkappa, \ldots, n) \tag{3.15}$$

and, for $\varkappa = -1$, an additional relation

$$a_0 = b_0 - \sum_{k=1}^{n} c_{0k} a_k$$

which is indispensable, if α_{-1} is to be computed by means of (3.13).

After solving the system (3.15) we can find ψ^* by means of (3.12), which can be represented by

$$\Psi^* = (E + \Gamma)\, Rf$$

$$\Gamma\psi = \int_{-1}^{1} \gamma(x, t)\, w(t)\, \psi(t)\, dt, \qquad \gamma(x, t) = -(-1)^{\lambda+\varkappa} 2^{\varkappa} \sum_{i,\,k=-\varkappa}^{n} \frac{\Delta_{ki}}{\Delta} N_i(t)\, P_{k+\varkappa}^{(\alpha,\,\beta)}(x) \tag{3.16}$$

where E is an identity operator, Δ is the determinant of (3.15) and Δ_{ki} is the algebraic co-minor of the corresponding element.

It should be noted that if $f(x)$ satisfies the condition (1.2) with $\mu > \operatorname{Re} \alpha$, and

$\mu > \mathrm{Re}\,\beta$, then Rf (and consequently ψ^*) will satisfy the H condition. Indeed, let us put

$$f(x) = f_1(x) + {}^1/_2\,[(1+x)f(1) + (1-x)f(-1)]$$

The ratio $f_1(x)/w(x)$ satisfies the H condition and is equal to zero at the ends of the segment $[-1, +]$. By (2.8), R operating on the second term will result in a polynomial, and Rf will, by the theorem of Plemel' – Privalov, satisfy the H condition.

In the discussion that follows, we shall assume the above statements to be valid.

4. Let H be a functional space satisfying the condition $H(\mu)$, $\mu > \mathrm{Re}\,\alpha$, $\mu > \mathrm{Re}\,\beta$ on the interval $[-1, +1]$, with the norm

$$\|\psi\|_H = \max|\psi| + \sup\frac{|\psi(t) - \psi(x)|}{|t-x|^{\mu}}$$

and let C be a space of continuous functions. We shall assume that

$$R \in [H \to C], \quad \Gamma \in [C \to C], \quad k, K \in [C \to H]$$

Also let U be a functional defined by (1.6) on the space H. The solution of (3.5) will be represented by

$$\psi^* = (E + \Gamma)\,R\,(f + K\psi^* - k\psi^*)$$

and the difference between the solutions of the exact (3.5) and the approximate (3.8) equation, will be

$$\psi^* - \Psi^* = (E + \Gamma)R\,(K\psi^* - k\psi^*)$$

Finally, let the kernels $k(x, t)$ and $K(x, t)$ be similar in the sense that for any $\psi \in C$

$$\|k\psi - K\psi\|_H \leqslant \eta\,\|\psi\|_C \tag{4.1}$$

Then

$$\|\psi^* - \Psi^*\|_C \leqslant \eta\,\|E + \Gamma\|\,\|R\|\,\|\psi^*\|_C = p\,\|\psi^*\|_C \tag{4.2}$$

or, in other words,

$$\|\psi^* - \Psi^*\|_C \leqslant \frac{p}{1-p}\|\Psi^*\|_C \qquad \text{for} \quad p = \eta\,\|E + \Gamma\|\,\|R\| < 1 \tag{4.3}$$

From (3.7) and (3.10) we obtain

$$\alpha_{-1} = \frac{1}{h_0^{(-\alpha,\,-\beta)}}\,U\,(K\Psi^* - k\psi^*)$$

which, together with (4.3), gives the following estimate

$$|\alpha_{-1}| \leqslant \frac{\|U\|}{|h_0^{(-\alpha,\,-\beta)}|}\,\frac{\eta + p\,\|k\|}{1-p}\,\|\Psi^*\|_C \tag{4.4}$$

Let us now estimate the norms of the operators appearing in the above relations. Using (2.8), we can write

$$Rf = (-1)^{\lambda+\varkappa}2^{\varkappa}P_{\varkappa}^{(\alpha,\,\beta)}(x)\,f(x) - \frac{b}{\pi i}\int_{-1}^{1}\frac{f(t) - f(x)}{t-x}\,\frac{dt}{w(t)}$$

from which,

$$\|Rf\|_C \leqslant A\max|f(x)| + B\sup\frac{|f(t) - f(x)|}{|t-x|^{\mu}} \leqslant \max\{A, B\}\,\|f(x)\|_H$$

$$A = 2^{\varkappa}\max|P_{\varkappa}^{(\alpha,\,\beta)}(x)|, \quad B = \frac{|b|}{\pi}\max\int_{-1}^{1}\frac{|t-x|^{\mu-1}}{|w(t)|}\,dt, \quad \|R\| \leqslant \max\{A, B\} \tag{4.5}$$

follows. Further $\|E+\Gamma\| \leqslant 1+\|\Gamma\|$, and it can be shown, that

$$\|\Gamma\| = \max \int_{-1}^{1} |\gamma(x, t) w(t)| \, dt \tag{4.6}$$

Let us introduce the following notation

$$k(x, t) - K(x, t) = \delta(x, t) \in H(\mu, \nu)$$

then,

$$\begin{aligned} \|\delta(x, t)\|_H = \max|\delta(x, t)| + \max \sup_{x, y} \frac{|\delta(x, t) - \delta(y, t)|}{|x-y|^{\mu}} + \\ + \max \sup_{t, s} \frac{|\delta(x, t) - \delta(x, s)|}{|t-s|^{\nu}} \end{aligned} \tag{4.7}$$

and, for any function $\psi \in C$

$$\|k\psi - K\psi\|_H \leqslant M \|\psi(x)\|_C \|\delta(x, t)\|_H$$

$$M = \max\left\{\int_{-1}^{1} |w(t)| \, dt, \quad \max \int_{-1}^{1} |t-s|^{\nu} |w(t)| \, dt\right\}$$

Hence, in (4.1) we can put

$$\eta = M \|\delta(x, t)\|_H \tag{4.8}$$

Analogously

$$\|k\| \leqslant M \|k(x, t)\|_H \tag{4.9}$$

Finally, we shall give the estimate for the norm of the functional U, which is

$$\|U\| \leqslant \max\left\{|h_0^{(-\alpha, -\beta)}|, \ \max \int_{-1}^{1} \frac{|t-x|^{\mu}}{|w(t)|} \, dt\right\} \tag{4.10}$$

5. Let us now take a part of the series of Jacobi polynomials of the function $k(x, t)$, as $K(x, t)$

$$k(x, t) \sim \sum_{k=0}^{\infty} N_k(t) P_k^{(-\alpha, -\beta)}(x), \qquad N_k(t) = \frac{1}{h_k^{(-\alpha, -\beta)}} \int_{-1}^{1} k(x, t) \frac{P^{(-\alpha, -\beta)}(x)}{w(x)} \, dx$$

$$(k = 0, 1, 2, \ldots)$$

From (3.13) we obtain

$$\begin{aligned} c_{ik} &= \frac{(-1)^{\lambda+\varkappa} 2^{\varkappa}}{h_i^{(-\alpha, -\beta)}} \int_{-1}^{1}\int_{-1}^{1} k(x, t) \frac{P_i^{(-\alpha, -\beta)}(x)}{w(x)} P_{k+\varkappa}^{(\alpha, \beta)}(t) w(t) \, dx \, dt \\ b_i &= \frac{1}{h_i^{(-\alpha, -\beta)}} \int_{-1}^{1}\int_{-1}^{1} k(x, t) \frac{P_i^{(-\alpha, -\beta)}(x)}{w(x)} Rf(t) w(t) \, dx \, dt \quad (i, k = -\varkappa, -\varkappa+1, \ldots, n) \end{aligned} \tag{5.1}$$

If a is a real number and b an imaginary number, then in (1.3) we have $\rho = 1$. Consequently, α and β are real, and $w(x)$ is positive. In this case all the roots of the polynomials $P_n^{(\alpha,\beta)}(x)$, and $P_n^{(-\alpha,-\beta)}(x)$ are real, simple and lie on the interval $(-1, +1)$, hence the Gauss – Jacobi formula [6] can be used in calculating the integrals (5.1).

The author expresses his gratitude to G.N. Pykhteev for discussing this work.

BIBLIOGRAPHY

1. Akhiezer N.I., O nekotorykh formulakh obrashcheniia singuliarnykh integralov (On some inversion formulas for singular integrals). Izv. Akad. Nauk SSSR. Ser. matem., Vol. 9, No. 4, 1945.

2. Ivanov V.V., Priblizhennoe reshenie singuliarnykh integral'nykh uravnenii v sluchae razomknutykh konturov integrirovaniia (Approximate solution of singular integral equations in case of open contours of integration) Dokl. AN SSSR, Vol. 111, No. 5, 1956.

3. Kalandiia A.I., O priblizhennom reshenii odnogo klassa singuliarnykh integral'nykh uravnenii (On the approximate solution of a class of singular integral equations). Dokl. AN SSSR, Vol. 125, No. 4, 1959.

4. Muskhelishvili N.I., Singuliarnye integral'nye uravneniia (Singular Integral Equations). Fizmatgiz, 1962.

5. Erdélyi A. ed., Higher Transcendental functions, Vol. 2, McGraw-Hill, 1953.

6. Sege G., Ortogonal'nye mnogochleny (Orthogonal Polynomials). Fizmatgiz, 1962.

Translated by L.K.